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The Nonlinear Schrödinger system: Collapse, Nonlinear damping, Noise, Impurities and Nonlocal dispersion

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Abstract

The paper reports on the effects of adding noise, damping and impurities to the 2-dimensional nonlinear Schrödinger equation. The effects of including long-range dispersion in the 1-dimensional nonlinear Schrödinger equation are also examined. The framework of the reported results is the application of nonlinear Schrödinger systems in the study of molecular systems and organic thin films. The paper includes numerical as well as analytical results.

1. Introduction

We consider the Schrödinger system with cubic nonlinearity. This system models optical media, molecular thin films in the continuum limit, deep water waves, and many other physical systems which exhibit weak nonlinearity and strong dispersion. The effects that we shall discuss were observed in our earlier work [1–7]. A motivation for studying the two-dimensional nonlinear Schrödinger equation (NLS) with fluctuations and impurities is our wish to understand efficient energy transfer in J-aggregates (Scheibe-aggregates) [8, 9]. Our starting point is a two-dimensional Davydov model with nonlinear coupling between the exciton and phonon system, and with white noise in the phonon system.

In section 2 we derive a single equation for the exciton system with multiplicative colored noise and a nonlinear damping term. In the continuum limit, the collective coordinate approach indicates that an energy balance between energy input (from the noise term) and dissipation can be established. Thus this model may describe the state of thermal equilibrium in the molecular aggregate. The coherent exciton moving on the aggregate [8, 9] is modelled by the ground state solution to the 2-D nonlinear Schrödinger equation, and the lifetime has been related to the collapse time of the ground state [10]. For sufficiently strong nonlinearity the thermal fluctuations will slow down the collapse. As a result this life time increases with the variance of the fluctuations, i.e., the temperature.

In Section 3, we model the presence of acceptor molecules in the J-aggregate by an impurity term in the nonlinear Schrödinger equation. The resulting dynamics of the moving exciton in the neighborhood of the acceptor molecule is extremely complicated. Our brief discussion focuses on the persistence of a dipole-like stationary state in the presence of DC-and AC-driving.

Finally, in Section 4, the role of long-range interaction in the molecular system is investigated in the one-dimensional case. Here, the nonlinear Schrödinger equation is augmented by a non-local dispersion term. As a result stationary solutions can only exist in a bounded interval of the excitation number in contrast to the usual NLS. Moving solitons are shown to radiate. The faster the motion the stronger is the radiation.

2. Noise and Damping

Following the derivation given in Ref. [1], we start by assuming that the coupled exciton-phonon system can be described by the following pair of equations:

$$i\hbar\dot{\psi}_n + \sum_{n'} J_{nn'}\psi_{n'} + \chi u_n \psi_n = 0, \tag{2.1}$$

$$M\ddot{u}_n + M\lambda\dot{u}_n + M\omega_o^2 u_n - \chi|\psi_n|^2 = \eta_n(t). \tag{2.2}$$

Here ψ_n is the amplitude of the exciton wave function corresponding to site n and u_n represents the elastic degree of freedom at site n. Furthermore, $-J_{nn'}$ is the dipole–dipole interaction energy, χ is the exciton-phonon coupling constant, M is the molecular mass, λ is the damping coefficient, ω_0 is the Einstein frequency of each oscillator, and $\eta_n(t)$ is an external force acting on the phonon system. To describe the interaction of the phonon system with a thermal reservoir at temperature T, $\eta_n(t)$ is assumed to be Gaussian white noise with zero mean and with the autocorrelation function

$$\langle \eta_n(t)\eta_{n'}(t')\rangle = 2M\lambda k_B T\delta(t-t')\delta_{nn'}, \tag{2.3}$$

in accordance with the fluctuation-dissipation theorem ensuring thermal equilibrium.

In order to derive a single equation for the dynamics of the exciton system, we start by writing the solution to eq. (2.2) in the integral form

$$u_n(t) = u_n^{(0)}(t) + \frac{\int_0^t \mathrm{d}t'(\mathrm{e}^{S_+(t-t')} - \mathrm{e}^{S_-(t-t')})[\chi|\psi_n(t')|^2 + \eta_n(t')]}{M(S_+ - S_-)}, \tag{2.4}$$

where

$$S_\pm = -\frac{\lambda}{2} \pm \sqrt{\left(\frac{\lambda}{2}\right)^2 - \omega_0^2} \tag{2.5}$$

and

$$u_n^{(0)}(t) = \frac{1}{(S_+ - S_-)}\{(\lambda u_n(0) + \dot{u}_n(0))(\mathrm{e}^{S_+t} - \mathrm{e}^{S_-t}) + u_n(0)(S_+\mathrm{e}^{S_+t} - S_-\mathrm{e}^{S_-t})\}. \tag{2.6}$$

Here $u_n(0)$ and $\dot{u}_n(0)$ are the molecular displacements and velocities at the initial moment of time. From eq. (2.6) we observe that $u_n^{(0)}(t)$ will decay as $\mathrm{e}^{-(\lambda/2)t}$. Since we will be interested in times such that $t \gg 2/\lambda$ we can neglect this transient term in eq. (2.4). Furthermore, using repeated partial integration we can write

$$\int_0^t dt'\, e^{S_\pm(t-t')} |\psi_n(t')|^2$$

$$= \left[-e^{S_\pm(t-t')} \left(\frac{1}{S_\pm} |\psi_n(t')|^2 + \frac{1}{S_\pm^2} \frac{d}{dt'} (|\psi_n(t')|^2) \right) \right]_{t'=0}^{t'=t}$$

$$+ \frac{1}{S_\pm^2} \int_0^t dt'\, e^{S_\pm(t-t')} \frac{d}{dt'} (|\psi_n(t')|^2). \tag{2.7}$$

The last term in eq. (2.7) can be neglected if $|\psi_n(t)|^2$ is assumed to vary slowly enough with time compared with the lattice vibrations, i.e., if

$$\frac{\omega_0^2 - (\lambda/2)^2}{\omega_0^4} \frac{1}{|\psi_n(t)|^2} \frac{d^2}{dt^2} (|\psi_n(t)|^2) \ll 1. \tag{2.8}$$

Neglecting all exponentially decaying transient terms, we thus obtain an approximate expression for the molecular displacements from eqs (2.4), (2.5) and (2.7):

$$u_n(t) \simeq \frac{\chi}{M\omega_0^2} \left(|\psi_n(t)|^2 - \frac{\lambda}{\omega_0^2} \frac{d}{dt} (|\psi_n(t)|^2) \right) + \sigma_n(t). \tag{2.9}$$

Here $\sigma_n(t)$, defined as

$$\sigma_n(t) = \frac{1}{(S_+ - S_-)} \int_0^t dt' (e^{S_+(t-t')} - e^{S_-(t-t')}) \eta_n(t'), \tag{2.10}$$

is a new stochastic force described by noise which is not white, but strongly colored [1].

Introducing the expression eq. (2.9) for the molecular displacements into eq. (2.1), we immediately get the following equation involving only exciton variables:

$$i\hbar\dot{\psi}_n + \sum_{n'} J_{nn'} \psi_{n'} + V |\psi_n|^2 \psi_n$$

$$- V \frac{\lambda}{\omega_0^2} \psi_n \frac{d}{dt} (|\psi_n(t)|^2) + \chi\sigma_n(t)\psi_n = 0. \tag{2.11}$$

Here we have introduced the nonlinearity parameter V defined as

$$V = \frac{\chi^2}{M\omega_0^2}. \tag{2.12}$$

By comparison with the derivation by Bang *et al.* [1], the main difference is that we retain one more term in the expansion eq. (2.7). The result of this is the presence of the nonlinear damping term $-V(\lambda/\omega_0^2)\psi_n(d/dt)(|\psi_n(t)|^2)$ in the exciton eq. (2.11).

Making the additional assumption that ψ_n varies slowly in space and that only nearest-neighbor coupling J is of importance, in the continuum approximation for the continuous exciton field $\psi(x, y, t) \equiv e^{-4iJt/\hbar}\psi_n(t)/l$ we obtain:

$$i\hbar\psi_t + Jl^2\nabla^2\psi + Vl^2|\psi|^2\psi - V\frac{\lambda}{\omega_0^2} l^2\psi(|\psi|^2)_t + \chi l^2\sigma\psi = 0. \tag{2.13}$$

Here, l is the distance between nearest neighbors, and $\sigma(x, y, t) = \sigma_n(t)/l^2$ is the noise density. Eq. (2.13) can be cast into a more convenient form by transforming into dimensionless variables,

$$\frac{x}{l} \to x, \quad \frac{y}{l} \to y, \quad Jt/\hbar \to t, \quad \sqrt{\frac{Vl^2}{J}}\,\psi \to \psi, \quad \frac{\chi l^2}{J}\,\sigma \to \sigma. \tag{2.14}$$

This leads to

$$i\psi_t + \nabla^2\psi + |\psi|^2\psi - \Lambda\psi(|\psi|^2)_t + \sigma\psi = 0, \tag{2.15}$$

where the nonlinear damping parameter Λ is given by

$$\Lambda = \frac{\lambda J}{\hbar\omega_0^2}. \tag{2.16}$$

It can easily be shown that in spite of the presence of the nonlinear damping and multiplicative noise terms in eq. (2.15), the norm, defined as

$$N = \int\int |\psi(x, y, t)|^2\, dx\, dy \tag{2.17}$$

will still be a conserved quantity, having the value $N = V/J$ if the exciton wave function is assumed to be normalized in the physical coordinates. By writing $\psi = \sqrt{n}\, e^{i\theta}$, the following equations for the amplitude and phase of the solution can be obtained from eq. (2.15)

$$\frac{1}{2} n_t + \nabla \cdot (n\nabla\theta) = 0,$$

$$-\theta_t - \Lambda n_t - (\nabla\theta)^2 + n + \frac{1}{\sqrt{n}} \nabla^2(\sqrt{n}) + \sigma(x, y, t) = 0. \tag{2.18}$$

The norm conservation is immediately seen from the first of these equations, while the second equation shows that the role of the damping term is to destroy the phase coherence of the solution and cause a diffusion-like behavior for the phase. The ordinary NLS Hamiltonian, defined as

$$H = \int\int \left(|\nabla\psi(x, y, t)|^2 - \frac{1}{2} |\psi(x, y, t)|^4 \right) dx\, dy, \tag{2.19}$$

will then in general no longer be conserved. Instead we find that

$$\frac{dH}{dt} = \int\int \sigma(x, y, t)(|\psi|^2)_t\, dx\, dy - \Lambda \int\int ((|\psi|^2)_t)^2\, dx\, dy. \tag{2.20}$$

Thus, the two terms provide energy input and energy dissipation to the exciton system, making an energy balance possible. Consequently, there is a possibility for the system to reach thermal equilibrium (see [4] for further discussion).

To investigate the influence on the collapse process of the damping and noise terms in eq. (2.15), we will use the method of collective coordinates. To this end a number of simplifying assumptions are appropriate. We assume isotropy, which effectively reduces the problem to one space dimension with the radial coordinate $r = \sqrt{x^2 + y^2}$. We also assume that the noise σ can be approximated by radially isotropic Gaussian white noise with autocorrelation function

$$\langle \sigma(r, t)\sigma(r', t') \rangle = \frac{D_r}{r} \delta(r - r')\delta(t - t'), \tag{2.21}$$

where D_r is the dimensionless noise variance. The validity of this approximation was discussed in Ref. [1]. Finally, we

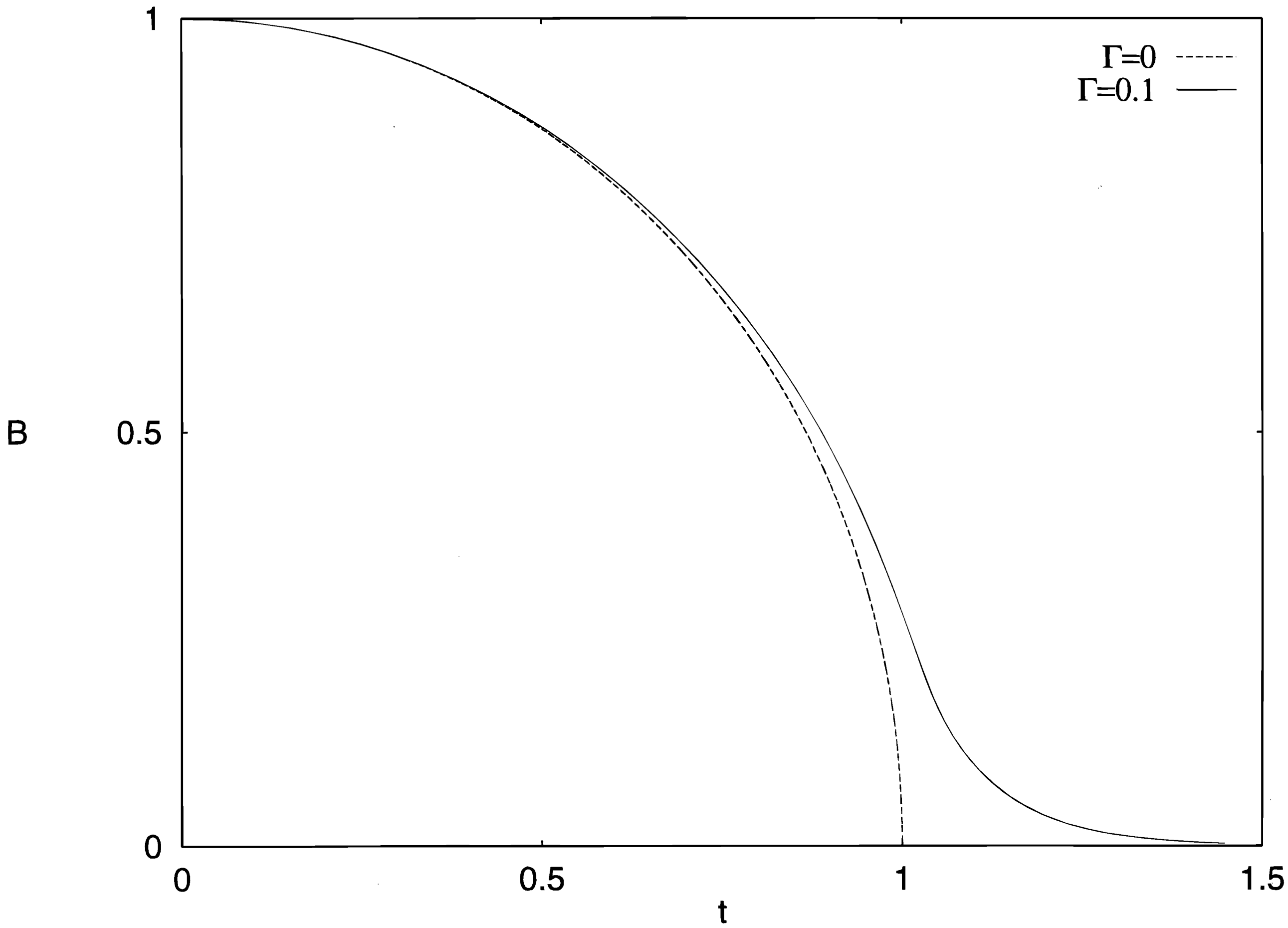


Fig. 1. Width, B, as function of time t in the absence of noise. Dashed line shows the analytical solution eq. (2.34) for $\Gamma = 0$ with $B_0 = |\Delta| = 1$; solid line the numerical solution of eq. (2.31) for $\Gamma = 0.1$. From [4].

assume that the collapse process can be described in terms of collective coordinates using the following self-similar trial function for the exciton wave function $\psi(r, t)$:

$$\psi(r, t) = A(t) \operatorname{sech}\left(\frac{r}{B(t)}\right) e^{i\alpha(t)r^2}. \tag{2.22}$$

With its three real time-dependent parameters A, B, and α determining the amplitude, width, and phase of the wave function, this trial function was used in Refs [2, 3] to investigate the case when $\Lambda = 0$ in eq. (2.15). The choice of this particular type of trial function can be motivated by regarding it as a generalization of the approximate ground state solution to the ordinary 2D NLS found by Anderson *et al.* [11]. From the definition (2.17) of the norm, we immediately obtain the relation between amplitude and width,

$$A(t) = \frac{\sqrt{N/s_{1,2}}}{B(t)}, \tag{2.23}$$

where the coefficient $s_{1,2}$ is obtained from the general definition

$$s_{m,n} = 2\pi \int_0^\infty r^m \operatorname{sech}^n(r)\, dr. \tag{2.24}$$

In [3] a variational approach was used, and from the Euler-Lagrange equations the relation

$$\alpha(t) = \frac{\dot{B}(t)}{4B(t)} \tag{2.25}$$

was obtained together with an ordinary differential equation for $B(t)$. However, in the presence of damping ($\Lambda \neq 0$) this technique is not applicable.

Instead, applying the virial theorem we will use the trial function eq. (2.22) with $\alpha(t)$ given by eq. (2.25) to derive an ordinary differential equation for $B(t)$. Defining the virial coefficient W as

$$W(t) \equiv \int \mathbf{r}^2 |\psi(\mathbf{r}, t)|^2 \, d\mathbf{r} = 2\pi \int_0^\infty r^3 |\psi(r, t)|^2 \, dr, \tag{2.26}$$

using eq. (2.15) we see that it satisfies the equation

$$\frac{1}{4}\frac{d^2W}{dt^2} = 2H - 2\pi\Lambda \int_0^\infty r^2 |\psi|^2 \partial_{tr}^2 |\psi|^2 \, dr + 2\pi \int_0^\infty r^2 |\psi|^2 \partial_r \sigma \, dr \tag{2.27}$$

where H is the Hamiltonian eq. (2.19). Using eqs (2.22–2.25), we arrive at the following differential equation for the width B of the exciton wave function:

$$B^3 \ddot{B} = \Delta - \Gamma \frac{\dot{B}}{B} - \frac{8\pi}{s_{3,2}} \int_0^\infty \left(1 - \frac{r}{B}\tanh\left(\frac{r}{B}\right)\right) \times \operatorname{sech}^2\left(\frac{r}{B}\right) \sigma r \, dr, \tag{2.28}$$

where the constants Δ and Γ are defined as

$$\Delta = \frac{4}{s_{3,2}}\left(s_{1,2} - s_{1,4} - \frac{Ns_{1,4}}{2s_{1,2}}\right), \tag{2.29}$$

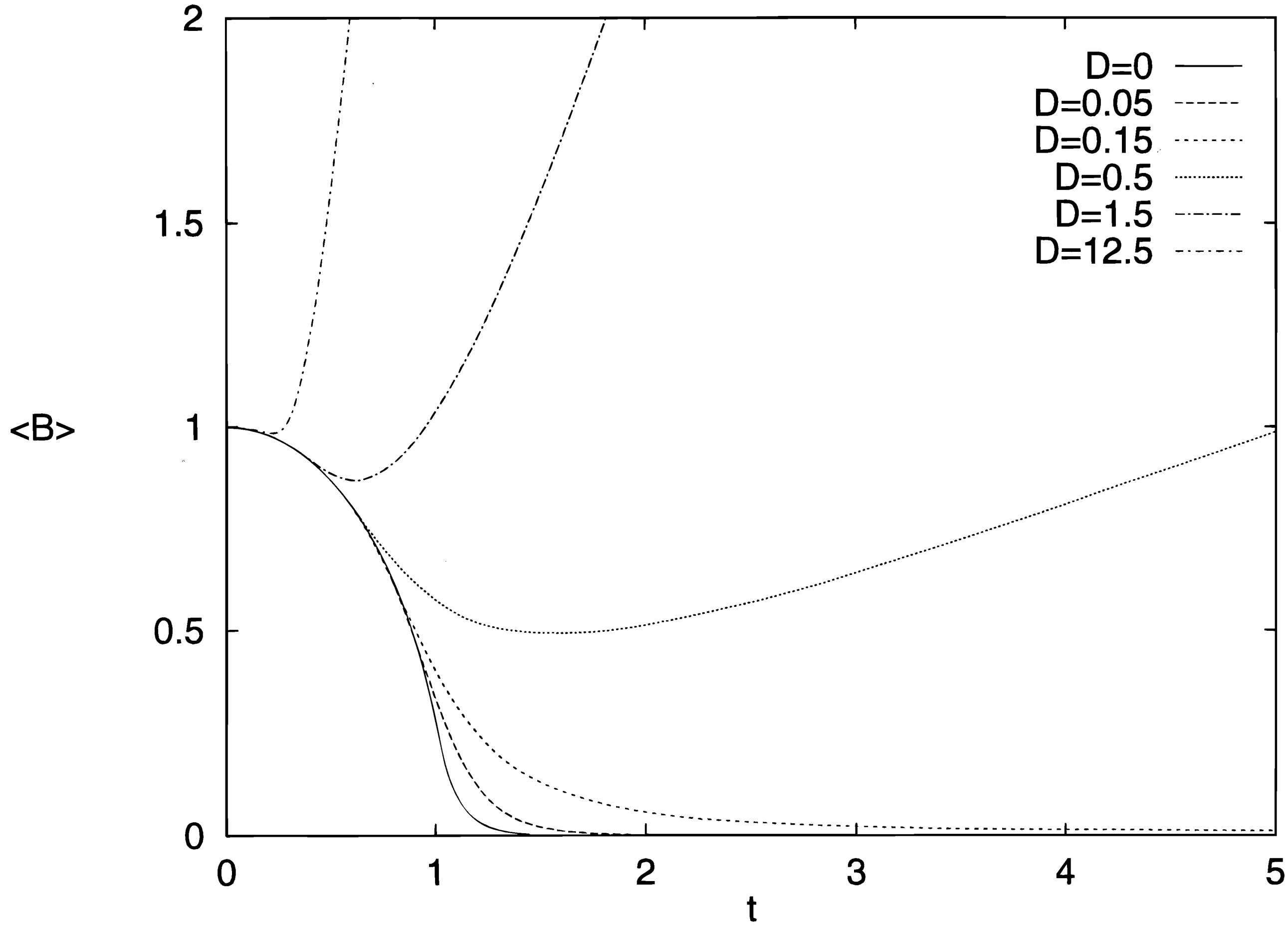


Fig. 2. Ensemble average of the width, $\langle B \rangle$, as a function of time t for different noise variances. From bottom to top we have $D = 0, 0.05, 0.15, 0.5, 1.5$, and 12.5, respectively. In all cases $B_0 = |\Delta| = 1$ and $\Gamma = 0.1$. From [4].

and

$$\Gamma = \frac{8N\Lambda}{s_{1,2}s_{3,2}}(s_{3,4} - s_{3,6}), \tag{2.30}$$

respectively. Note that Δ and Γ depend on the initial conditions via N, and that, while Δ can be either positive or negative, Γ is always positive. In the absence of noise and damping, it is evident from eq. (2.28) that collapse will occur if and only if $\Delta < 0$.

In analogy with the treatment for the undamped case in Ref. [3], we find that it is possible to transform eq. (2.28) into a simpler stochastic differential equation without changing the Fokker-Planck equation for the system. This equation, which therefore gives an equivalent description of the process, attains the form

$$\ddot{B} = \frac{\Delta}{B^3} - \frac{\Gamma \dot{B}}{B^4} + \frac{h(t)}{B^2}, \tag{2.31}$$

where $h(t)$ is white noise with the autocorrelation

$$\langle h(t)h(t') \rangle = 2D\delta(t - t'). \tag{2.32}$$

The parameter D, giving the variance of $h(t)$, is related to the variance D_r of $\sigma(r, t)$ defined in eq. (2.21) through

$$D = \frac{32\pi^2 D_r}{s_{3,2}^2}(s_{3,4} - s_{3,6}). \tag{2.33}$$

We now analyse the influence of damping and noise on the collapse process by solving the ordinary stochastic differential equation (2.31) for the width B of the trial function (2.22) numerically. A detailed description of the numerical approach can be found in [4]

With neither damping nor noise in the system ($\Gamma = D = 0$) and $\Delta < 0$, the well-known exact solution

$$B(t) = B_0\sqrt{1 - \frac{t^2}{t_c^2}}, \quad t_c \equiv \frac{B_0^2}{\sqrt{|\Delta|}}, \tag{2.34}$$

fulfilling the initial conditions

$$B(0) = B_0, \quad \dot{B}(0) = 0 \tag{2.35}$$

is easily obtained. Thus, the solution collapses and ceases to exist at the collapse time $t = t_c$. When the damping term is present in eq. (2.31) we find that strictly speaking no collapse will occur, since the solution will be well-defined for all t. Instead we find that $B(t)$ will approach zero exponentially for large t,

$$B(t) \sim e^{-(|\Delta|/\Gamma)t}, \quad t \to \infty, \tag{2.36}$$

so that the process can be considered as a collapse process with an infinite collapse time. This type of behavior will be called "pseudo-collapse". The difference between the damped and the undamped cases is illustrated in Fig. 1. As can be seen, the initial stages of the pseudo-collapse process

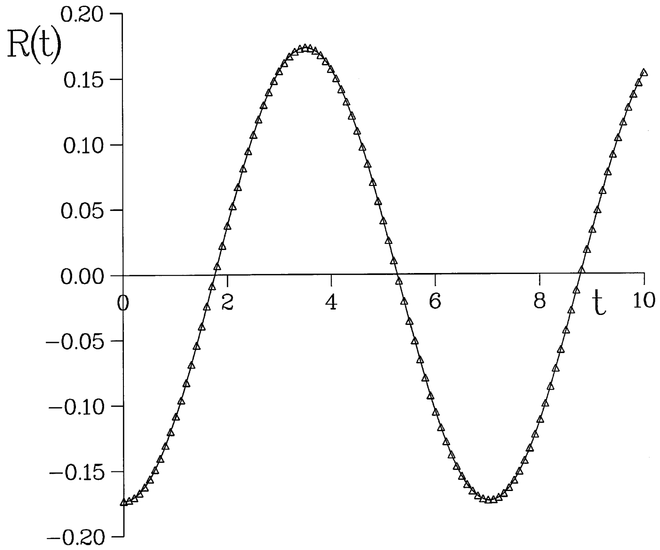


Fig. 3. The center of mass motion for a soliton obtained by direct numerical solution of Eq. (3.3) (solid line) and by solution of the separated equation of motion (indicated by triangles). From [6].

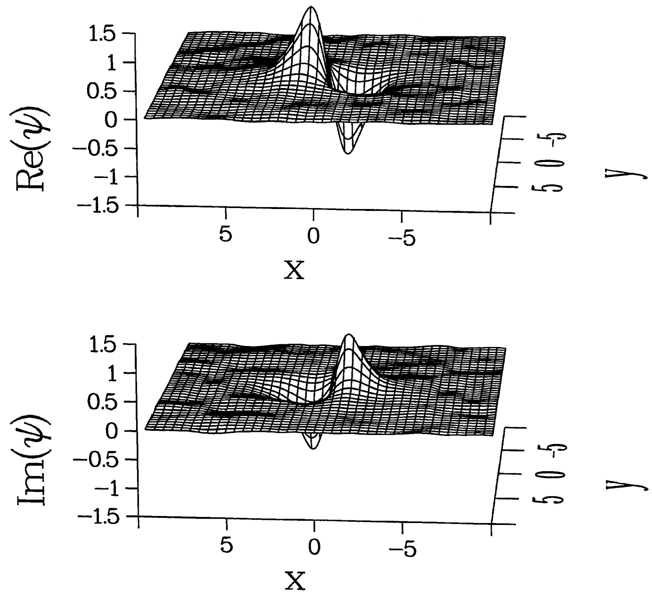


Fig. 5. The real and imaginary parts of a dipole-like stationary state $\psi(r, t)$ at an impurity. From [6].

will resemble a pure collapse as long as the damping is small. One can then roughly define a "pseudo-collapse-time" as the time where the asymptotic behavior (2.36) sets in.

To illustrate how the noise affects the (pseudo-)collapse process, we show the behavior of $\langle B(t)\rangle$ in Fig. 2 for the parameter values $B_0 = |\Delta| = 1$, $\Gamma = 0.1$, and different values of the noise variance. It can be seen that for $D < D_{\text{crit}} \simeq 0.15$, the effect of the noise is to delay the pseudo-collapse in terms of the ensemble average of the width, in analogy with the similar result obtained in Ref. [2] for the undamped case. For $D > D_{\text{crit}}$, we observe a non-monotonic behavior of $\langle B(t)\rangle$. Initially, the average width will decrease in a similar way as when $D < D_{\text{crit}}$, but after some time $\langle B(t)\rangle$ will reach a minimum value and diverge as $t \to \infty$. This is due to the fact that for $D > D_{\text{crit}}$, the noise is strong enough to destroy the pseudo-collapse and cause dispersion for some of the systems in the ensemble. As $t \to \infty$ the dominating contribution to $\langle B(t)\rangle$ will come from the dispersing systems for which $B \to \infty$, and consequently $\langle B(t)\rangle$ will diverge for $D > D_{\text{crit}}$. As can be seen in Fig. 2 the minimum value of $\langle B(t)\rangle$ will increase towards B_0 as D increases.

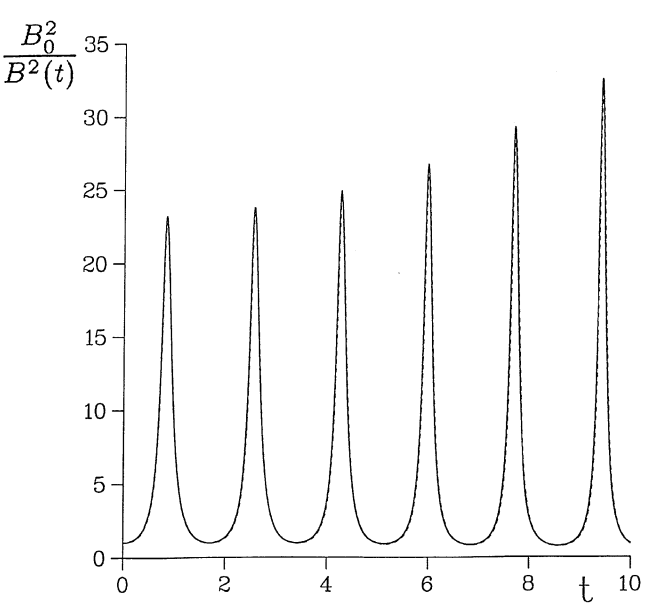


Fig. 4. The width of the soliton is $B(t)$, $B_0 \equiv B(0)$. Numerical solution of eq. (3.3) using (3.5) (solid line) compared to analytical approach (dashed line). From [6].

3. Impurity model

In this section we deal with soliton dynamics in the close vicinity of an impurity. We consider the impurity potential in the parabolic approximation and show that in this case the center of mass motion and the internal motion decouple, and using the lens transformation [6] one can reduce the problem to an Ermakov–Pinney differential equation [12].

As in the previous section we restrict ourselves to the continuum limit. In this limit the problem may be approximated by the two-dimensional nonlinear Schrodinger equation with a spatially variable coefficient

$$i\hbar\psi_t + Jl^2\nabla^2\psi + Vl^2|\psi|^2\psi = E(\mathbf{r})\psi, \tag{3.1}$$

where, for the sake of simplicity, we shall assume that $E(\mathbf{r})$ is an axially symmetric Gaussian function

$$E(r) = E\exp(-r^2/r_0^2), \tag{3.2}$$

where E is the strength of the impurity and r_0 is its radius. Introducing dimensionless variables $r/r_0 \to r$, $t/(r_0^2\hbar/l^2J) \to t$, $\psi/\sqrt{J/(Vr_0^2)} \to \psi$, and $E/(Jl^2/r_0^2) \to E$ we obtain

$$i\psi_t + \nabla_r^2\psi + |\psi|^2\psi = E(1 - r^2)\psi \tag{3.3}$$

in the close vicinity of the impurity, since the impurity potential $E\exp(-r^2) \simeq E(1 - r^2)$. In this parabolic case the motion of the center of mass of a soliton, $R(t)$, and internal degrees of freedom are separated, even when $E = E(t)$. This case is investigated in details in [6] using results for the so-called Ermakov-Pinney equations. For combined DC

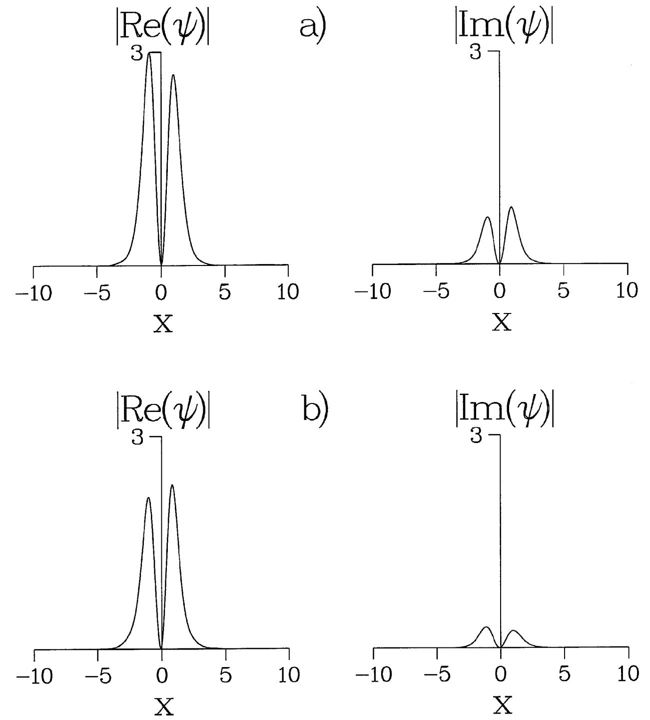


Fig. 6. Relative oscillations of the absolute value of the real and imaginary parts of the stationary state $\psi(x, 0, t)$, at times $t = 30$ (a) and 31 (b). From [6].

and AC dependence

$$E(t) = -\frac{\Omega_0^2}{4}(1 + \lambda \cos \Omega t) \tag{3.4}$$

we get the center of mass motion shown in Fig. 3 for $\Omega_0 = 1.7$, $\Omega = 2\Omega_0$, and $\lambda = 0.1$. In this case of parametric resonance, the minimal width of the soliton $B(t)$, obtained from the numerical solution of eq. (3.3) by the relation

$$B(t) \propto \left[\int d\mathbf{r}\, |\psi(\mathbf{r}, t)|^4\right]^{-1} \tag{3.5}$$

decreases rapidly as seen in Fig. 4. The agreement between numerical and analytical results is good.

The soliton dynamics in the presence of the impurity is very complicated. Thus a dipole-like nonlinear excitation exists as the stationary state shown in Fig. 5. Here the distribution of the real and imaginary parts of the wave function $\psi(\mathbf{r}, t)$ are depicted for $E = 5.0$, the norm $N = 11.7$, and $t = 40$. The center of mass is constantly located at the impurity at $(x, y) = (0, 0)$. The oscillatory nature of the stationary solution is illustrated in Fig. 6. For $N = 11.7$ this state exists for $E > E_{\text{th}} \simeq 4.5$. For larger values of N of the initial data, the threshold value for E, E_{th}, also increases. This complicated dynamics can also be analysed by applying the method of collective coordinates [6, 7]. We use the test function

$$\psi(\mathbf{r}, t) = e^{i\mu t}\left[A_1(t)\, e^{i\delta(t)/2} F\left(\left|\frac{\mathbf{r} - \boldsymbol{\rho}}{B}\right|\right) - A_2(t)\, e^{-i\delta(t)/2} F\left(\left|\frac{\mathbf{r} + \boldsymbol{\rho}}{B}\right|\right)\right], \tag{3.6}$$

where μ is the nonlinear frequency. $\boldsymbol{\rho}$ and $-\boldsymbol{\rho}$ are the positions of the extrema of the excitation, B is the common width of the pulses, and $A_1(t)$, $A_2(t)$, and $\delta(t)$ are the time varying amplitudes and phase difference, respectively. As common profile function, $F(z)$, for the two pulses the Gaussian $F(x) \equiv \exp(-x^2/2)$ is used. Figure 7 shows the resulting width, B, and the distance, ρ, between the extrema. Because of the presence of the impurity this oscillation persists for normalized norm $n \equiv N/8\pi \leqslant 1$. The predictions are in good agreement with numerical solution of eq. (3.3).

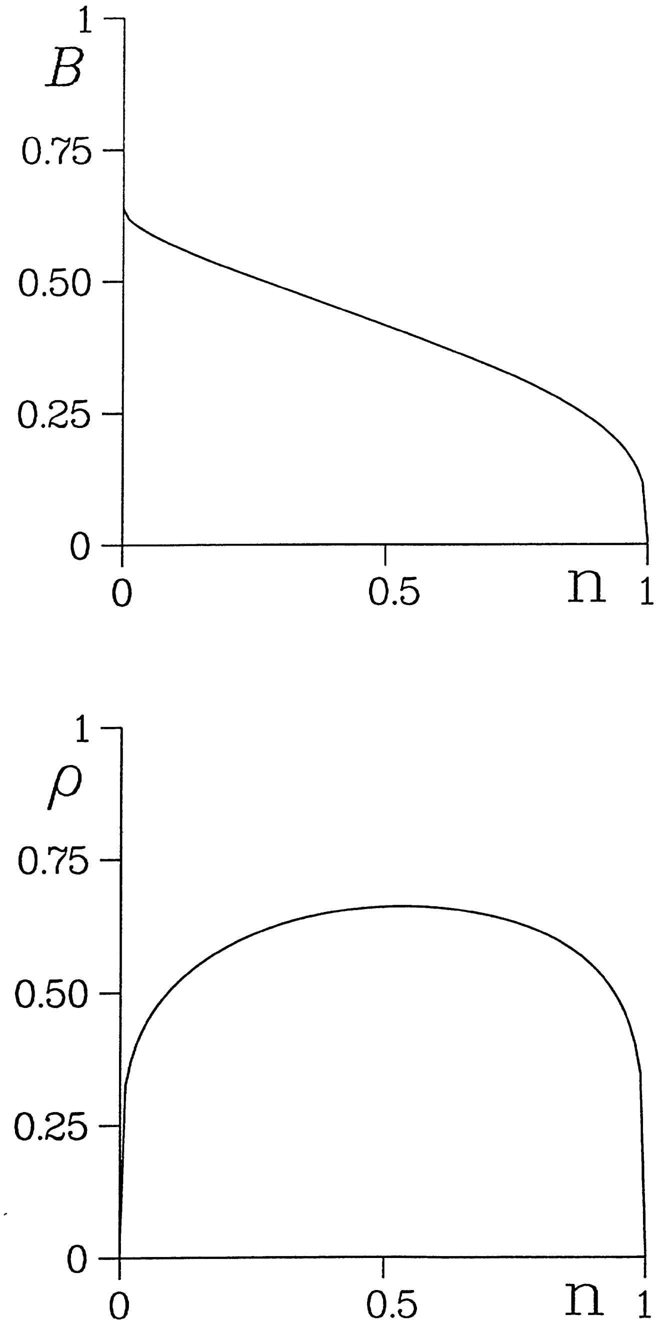


Fig. 7. Predictions obtained by method of collective coordinates, using the test function given by eq. (3.6). Upper part: width, B, versus normalized number n. Lower part: Half distance, ρ, between extrema of the stationary dipole-like excitations versus n. From [6].

4. Nonlocal dispersion

In this section we consider the dynamics of self-interacting quasi-particles in one-dimensional systems with long-range dispersive interactions. We take the matrix element of transfer, $J(x)$, to be of the Kac-Baker form $J(x) = J \exp(-\alpha|x|)$ with α being the inverse radius of interaction and J its intensity. The equation of motion for the excitation wave function then becomes the nonlocal nonlinear Schrödinger equation (Details are given in [5].)

$$i\partial_t \psi + \int_{-\infty}^{\infty} J(x-y)(\psi(y,t) - \psi(x,t))\,dy + |\psi|^2\psi = 0. \quad (4.1)$$

The kernel $J(x)$ in this integro-differential equation is the Green's function given by the equation $(\partial_x^2 - \alpha^2)J(x) = -2J\alpha\delta(x)$. As a consequence eq. (4.1) may be rewritten as

$$i\partial_t \psi + \frac{2J}{\alpha}\frac{\partial_x^2}{\alpha^2 - \partial_x^2}\psi + |\psi|^2\psi = 0. \quad (4.2)$$

The linear part of eq. (4.2) corresponds to the dispersion law $\omega(k) = (2J/\alpha)(k^2/(\alpha^2 + k^2))$ [5]. When the characteristic length scale of the excitations is large compared with the radius of the dispersive interaction (i.e., $\alpha \to \infty$, $J \to \infty$, $J\alpha^{-3} \to$ const.), eq. (4.2) reduces to the NLS equation. However, if the width of the excitations and the radius of interaction, α^{-1}, are of the same order, the nonlocal effects represented by the pseudo-differential operator in eq. (4.2) become important.

Rescaling variables $z = \alpha x$, $\tau = (2J/\alpha)t$, $\phi(z,\tau) = \sqrt{(\alpha/2J)}\,\psi(x,t)$, instead of eq. (4.2) we obtain

$$i\partial_\tau \phi + \frac{\partial_z^2}{1 - \partial_z^2}\phi + |\phi|^2\phi = 0, \quad (4.3)$$

and

$$\int_{-\infty}^{\infty} |\phi(z,\tau)|^2\,dz = \frac{\alpha^2}{2J}N \equiv \mathcal{N}, \quad (4.4)$$

where N, the excitation number, like the Hamiltonian H is an integral of motion. We look for stationary solutions to eq. (4.3) in the form

$$\phi = \frac{b}{\sqrt{b^2 - 1}}F(z,b)\exp(i\lambda^2\tau) \quad (4.5)$$

where λ is a spectral parameter and with $b = \lambda^{-1}\sqrt{\lambda^2 + 1}$ being the width of the solution. Here F must satisfy $(d^2/dz^2)(F - F^3) - Fb^{-2} + F^3 = 0$. Under the boundary conditions $F(\xi) \to 0$ for $\xi \to \pm\infty$ this equation has the solution

$$\exp\left(\frac{2z}{b}\right) = \frac{F_1 + F_2\mu}{F_1 - F_2\mu}\left(\frac{1-\mu}{1+\mu}\right)^{3/b} \quad (4.6)$$

where

$$F_{\frac{1}{2}}^2 = \frac{1}{b^2}\left(\frac{b^2+3}{4} \mp \sqrt{\left(\frac{b^2+3}{4}\right)^2 - b^2}\right)$$

and $\mu = (F_1^2 - F^2)^{1/2}(F_2^2 - F^2)^{-1/2}$. The solution given by eq. (4.6) exists only for $b \geqslant 3(\lambda^2 \leqslant \frac{1}{8})$. Introducing eq. (4.6) into eq. (4.4) we get

$$\mathcal{N} = \frac{1}{b^2 - 1}\left(3b + \frac{b^2 - 9}{8}\ln\left(\frac{b^2 + 4b + 3}{b^2 - 4b + 3}\right)\right). \quad (4.7)$$

Figure 8 shows that the stationary solution exists only for $\mathcal{N} \leqslant \mathcal{N}_{max} \simeq 1.127685$, or $N \leqslant 2.25537J\alpha^{-2}$. Thus, in contrast to the usual NLS equation which has stationary solutions for any excitation number, the nonlocal NLS equation has stationary solutions only in a finite interval of $\mathcal{N} \in [0, \mathcal{N}_{max}]$. $\mathcal{N}_{max}$ tends to infinity in the limit of the usual NLS. Figure 8 shows that, when the excitation number $\mathcal{N}$ varies in the interval $[1.125, \mathcal{N}_{max}]$, there are two values of λ for a given $\mathcal{N}$, i.e., two branches of stationary solutions exist in that interval. The first branch corresponds to the part of the $\mathcal{N}(\lambda)$ for which $d\mathcal{N}/d\lambda > 0$, and it exists for any $\mathcal{N} \leqslant \mathcal{N}_{max}$. For $\mathcal{N} \ll \mathcal{N}_{max}(\lambda \ll 1/\sqrt{8})$ the solution reduces to $F = \sqrt{2}b^{-1}\,\text{sech}\,(z/b)$ such that it coincides with the stationary solution of the usual NLS equation. The second branch with $d\mathcal{N}/d\lambda < 0$ exists only in the interval $[1.125, \mathcal{N}_{max}]$. At $\mathcal{N} = \mathcal{N}_{cusp} = 1.125$ we obtain from eq. (4.6) a cusp soliton of the form $\phi(z,\tau) = (3/8)^{1/2}\exp(i(\tau/8) - (1/3)|z|)$. A similar solution was first found in the theory of shallow water waves [13]. Recently such a solution was obtained by Alfimov *et al.* [14] as a static solution of the nonlocal Klein-Gordon equation. The needed stability investigation is carried out in detail in [5], and the result is that the stationary solution is unstable for $(d\mathcal{N}/d\lambda) < 0$, i.e., the solitons of the second branch, and in particular the cusp soliton, are unstable. Using a slightly perturbed stationary solution of the two branches as initial condition the numerical integration of eq. (4.3) shown in Fig. 9(a) and (b) exhibits oscillatory and blow-up behavior, respectively, as suggested by the analysis in [5]. The oscillatory behavior indicates stability of the soliton of the first branch.

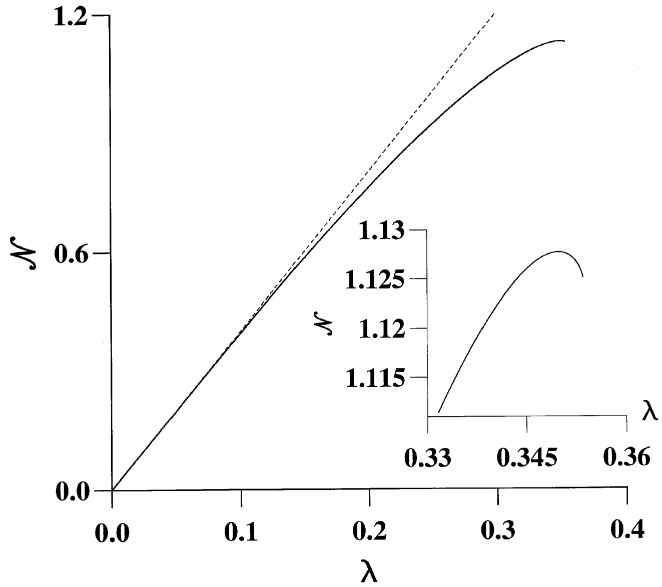


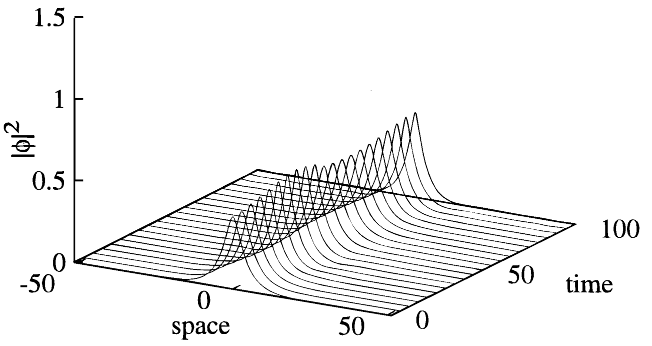
Fig. 8. Excitation number, $\mathcal{N}$, versus spectral parameter, λ, for the stationary solutions. Dashed line: NLS equation. Solid line: nonlocal NLS, eq. (4.3). The inset shows the two branches at the maximal excitation number. The endpoint corresponds to the cusp soliton. From [5].

Since the usual NLS equation is Galilean invariant the solitons can move without changing their shape and velocity. This is not the case for the nonlocal NLS equation. To investigate propagating solutions we introduce the moving frame of reference in which the center of mass of the excitation is at rest: $\xi = z - v\tau$, $\bar{\tau} = \tau$

$$\phi(z,\tau) = \varphi(\xi,\bar{\tau})\exp\left(\frac{i}{2}v\xi + i\lambda^2\bar{\tau} + \frac{i}{4}v^2\bar{\tau}\right), \quad (4.8)$$

where v is the velocity. Applying this set of transformations to eq. (4.3) we obtain for small velocities v

$$i\partial_{\bar{\tau}}\varphi - \frac{\lambda^2(1 - \partial_\xi^2) - (1 + iv\,\partial_\xi)\,\partial_\xi^2}{1 - iv\,\partial_\xi - \partial_\xi^2}\varphi + |\varphi|^2\varphi = 0. \quad (4.9)$$



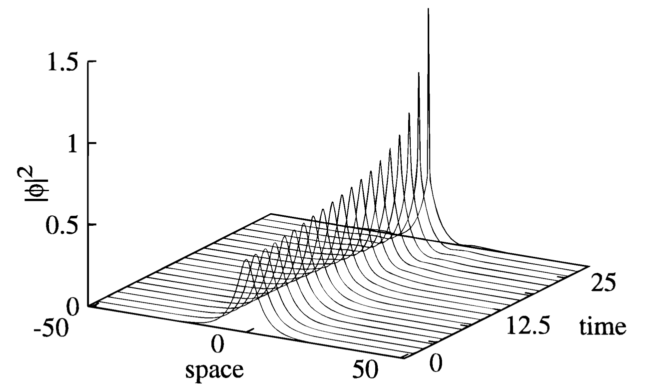


Fig. 9. Dynamical behavior of solutions to nonlocal NLS eq. (4.3) for initial conditions given by eq. (4.6) with slightly perturbed amplitude. (top) stable branch, $\lambda = 0.279$; (bottom) unstable branch, $\lambda = 0.352$. From [5].

Following Kuehl and Zhang [15] we rewrite eq. (4.9) using the Fourier transform $\varphi^F(k, \bar{\tau}) = \int_{-\infty}^{\infty} \exp(ik\xi)\varphi(\xi, \bar{\tau})\, d\xi$,

$$i\partial_{\bar{\tau}}\varphi^F(k, \bar{\tau}) - \omega_v(k)\varphi^F(k, \bar{\tau}) + NL(k, \bar{\tau}) = 0, \tag{4.10}$$

where $\omega_v(k) = [\lambda^2(1 + k^2) + (1 + vk)k^2]/[1 - vk + vk^2]$, the dispersion law in the moving frame of reference, and $NL(k, \bar{\tau}) \equiv \int_{-\infty}^{\infty} \exp(ik\xi)\,|\varphi(\xi, \bar{\tau})|^2\varphi(\xi, \bar{\tau})\, d\xi$ is introduced.

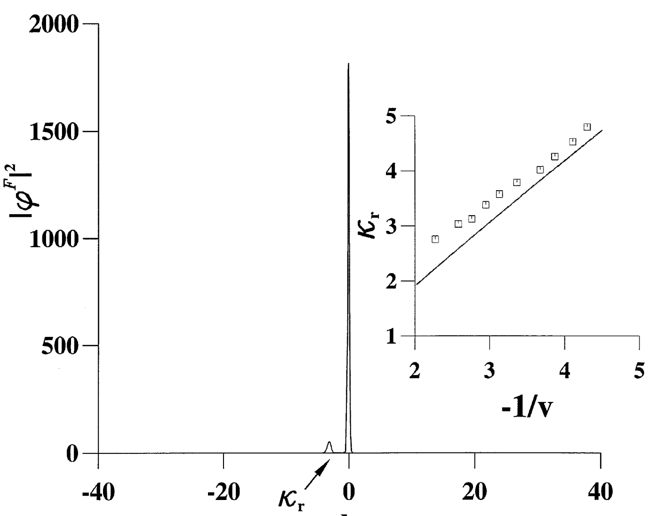


Fig. 10. Fourier spectrum of the moving soliton. Peak at $k = 0$ (soliton), $k = \kappa_r$ (radiation). Inset shows the resonant radiation wave number, κ_r, versus $-1/v$, v being the soliton velocity: Solid curve eq. (4.14); Squares, from numerical simulations of eq. (4.3). From [5].

The equation $\omega_v(k) = 0$ has the real root

$$k = k_r \simeq -\frac{\lambda^2 + 1}{v}$$

for small v. The existence of the root $k = k_r$ means [15, 16] that a wave with wavenumber k_r will be resonantly excited by a soliton, forming an oscillatory tail.

Introducing $\varphi(\xi, \bar{\tau}) = \phi_0(\xi) + f(\xi, \bar{\tau})$, where $\phi_0(\xi)\exp(i\lambda^2\bar{\tau})$ is the stationary solution given by eq. (4.5) and the function $f(\xi, \bar{\tau})$ describes the change of the shape of the soliton and the radiation, we get for the Fourier transform of $f(\xi, \bar{\tau})$

$$\begin{aligned} &i\partial_{\bar{\tau}} f^F - \omega_v(k) f^F \\ &= (\omega_v(k) - \omega_{v=0}(k))\phi_0^F(k) - \int_{-\infty}^{\infty} \exp(ik\xi)\{\phi_0^2(2f + f^*) \\ &\quad + \phi_0(f^2 + 2|f|^2) + |f|^2 f\}\, d\xi. \end{aligned} \tag{4.11}$$

In the resonant region $k \simeq k_r$, we neglect nonlinear terms in eq. (4.11) and use $\omega_v(k) \simeq v(k - k_r)$, $(\omega_v(k) - \omega_{v=0}(k))\phi_0^F(k_r) \simeq -(\lambda^2 + 1)\phi_0^F(k)$. Thus, returning to the real space $(\xi, \bar{\tau})$, we obtain

$$\begin{aligned} &i\,\partial_{\bar{\tau}} f - iv\,\partial_\xi f + (vk_r + 2\phi_0^2) f + \phi_0^2 f^* \\ &= -(\lambda^2 + 1)\phi_0^F(k_r)\delta(\xi). \end{aligned} \tag{4.12}$$

With the initial condition $f(\xi, \bar{\tau} = 0) = 0$, the solution of eq. (4.12) for $k_r \gg 1$ becomes

$$f(\xi, \bar{\tau}) = -ik_r\phi_0^F(k_r)\exp(-i\kappa_r\xi)(\theta(\xi + v\bar{\tau}) - \theta(\xi)), \tag{4.13}$$

where $\theta(z)$ is the Heaviside function and

$$\kappa_r = k_r + \frac{1}{v\xi}\int_0^\xi \phi_0^2(\xi')\, d\xi' \tag{4.14}$$

is the effective resonant wavenumber. Thus, a moving soliton stimulates radiation in the rear with a wavelength proportional to the velocity v. The amplitude of the radiation is proportional to $\phi_0^F(k_r)$ and for small values of λ (i.e., for small values of the excitation number $\mathcal{N}$) it decreases as $\exp(-(\pi/2\lambda v))$. In Fig. 10 the Fourier spectrum of the propagating soliton is shown. The numerical simulations of the dependence of the resonant wavenumber on the soliton velocity are in reasonable agreement with our analytical prediction.

5. Conclusions

The NLS systems with multiplicative noise, nonlinear damping, impurities and non-local dispersion exhibit a variety of interesting effects which may be useful for modelling the dynamical behavior of 1- and 2-dimensional systems. In this paper we have considered only the continuum approximation. Work on the corresponding discrete system is reported in Ref. [17].

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